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Singularity analysis, balance equations and soliton solution of the nonlocal complex Ginzburg-Landau equation

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Abstract. The modified complex Ginzburg-Landau equation (mCGLE) which includes a delayed response term in the integral form is analysed. In particular, a singularity analysis of mCGLE is presented. It is shown that this equation fails to pass the Painlevé test when the non-conservative terms are nonzero. Nevertheless, exact solutions to this equation do exist. Stationary solutions can be treated using the 'segment balance' method which is an extension of conservation laws to non-conservative systems. This method is used to derive an exact soliton solution of mCGLE.

Key words: solitons, integrability, singularity analysis, complex Ginzburg-Landau equation.

1. Background

Since the pioneering work of Painlevé, it has been known that information regarding the integrability of certain differential equations can be extracted from an analysis of the singularity structure of the equation. Thus, an equation satisfies the Painlevé condition if its general solution has no movable critical points. This can be checked by making a Laurent expansion of the solution near a movable singularity. For the equation to pass the test it is necessary that the leading-order exponent be an integer and that a recurrence relation can be found to relate all coefficients in the expansion. A general procedure is given in the book by Ablowitz and Clarkson [1, Section 7.2]. Interest in this type of analysis has increased in recent years due to applications of fully integrable equations in many areas of physics and engineering. The propagation of optical solitons in optical fibres is a good example of this.

Firstly, analysis was carried out on the KdV, mKdV and Burgers's equations [2], and then it was shown that the nonlinear Schrödinger equation (NLSE) passes the Painlevé test [3]. Work continued with two coupled NLSEs [4] and it was shown that the Heisenberg spin chain equation also has the property [5]. More recently, the complex Ginzburg-Landau equation (CGLE) has been considered, and although the Painlevé process shows that, as expected, the equation does not possess the property, the analysis is still useful in suggesting a suitable *ansatz* which can be used in finding solutions of the original equation [6, 7]. Some other equations related to the NLSE have been analyzed as well [8, 9]. They pass the test only for specific values of the parameters.

In this work, we study a modified CGLE with an additional integral term which is non-local in time. This term describes a memory in the system under consideration. One of the applications of this equation is to a passively mode-locked laser with slow saturable absorber in the cavity [10]. In this case the equation describes short-pulse generation by the laser. Other applications include pulse propagation in resonant media [11] where a delayed response function also occurs. We show that the equation doesn't satisfy the Painlevé property but suppose that the *ansatz*, which is similar to that in [7], may solve the equation. In addition, we suggest a different technique for finding solutions of the modified complex Ginzburg-Landau equation. It is based on balance equations which can be derived from this equation. When non-conservative terms are removed from the equation, the balance equations reduce to the usual conservation laws. On the other hand, with the non-conservative terms, the balance equations serve roles similar to conservation laws, and in some cases allow us to find solutions of the CGLE. This can be done for the CGLE in a general form with the terms which may be responsible for certain physical effects in optics and in other fields. Here we present an analysis for finding a soliton solution in the particular case of the CGLE with an integral term.

2. Modified Ginzburg-Landau equation

Many physical processes are governed by the Ginzburg-Landau equation [12, 13]. These include effects which relate to systems far from equilibrium [14]. For example, lumped effects are present in a ring laser system, but it can be modeled as a continuous system if the field changes only slightly on each round trip of the circuit. The pulse evolution is then described by a modified CGLE with nonlinear and non-conservative terms.

The normalized equation which we are studying here is

$$\mathrm{i}\psi_z + \tfrac{1}{2}D\psi_{tt} + |\psi|^2\psi = \mathrm{i}\delta\psi + \mathrm{i}\beta\psi_{tt} + \mathrm{i}\varepsilon|\psi|^2\psi + \mathrm{i}\alpha\psi\int_{-\infty}^{t}|\psi|^2\,\mathrm{d}t'. \tag{1}$$

This equation may be called the modified complex Ginzburg-Landau equation (mCGLE), and it now includes non-conservative, non-local and nonlinear terms. The last term on the right-hand side of (1), which involves memory of the previous values of the field, is non-local in t. The interpretation of the variables depends on the particular physical problem. In optics, z is the propagation distance or the cavity round-trip number (treated as a continuous variable), t is the retarded time, δ represents a constant gain or loss, β indicates band-limited gain (*e.g.* due to an EDFA, where the gain band may be about 30 nm around 1·5 microns), and ε is a nonlinear gain (or 2-photon absorption if negative). Clearly, δ would usually be the material attenuation and so would be negative. We do not specify that the coefficients be small, and so the analysis is not limited to the perturbation regime.

Particular cases of this equation have been studied in a number of publications related to various physical situations [11, 15–17]. An exact solution of Equation (1) without the integral term on the r.h.s. has been presented by Pereira and Stenflo [15]. An exact solution of Equation (1), without the cubic and spectral filtering terms on the r.h.s., has been presented by Grigoryan [16] and investigated numerically in [17]. Some analysis of the equation which is more general than Equation (1) has been given by Grigoryan and Muradyan [11]. On the other hand, the particular case (1) is important in itself, and we concentrate on it here.

3. Painlevé analysis

Any solution of an equation must be in accord with the singularity structure of that equation. A tool for investigating that structure is the Painlevé analysis. This analysis can be applied both for ODEs and PDEs [1]. The power of the Painlevé test lies in its easy algorithmic implementability. The main requirement is the representation of any possible solution in the form a Laurent expansion in the neighborhood of a movable singularity:

$$\psi = \chi(z,t)^{-p} \sum_{j=0}^{\infty} u_j(z)\chi(z,t)^j,$$

where p is the leading-order exponent, $\chi(z,t)$ is the expansion variable, and $u_j(z)$ is a set of analytic functions of z.

There are two necessary conditions for an ODE to pass the Painlevé test:

(1) the leading-order α must be an integer, and
(2) it must be possible to solve the recursion relation for the coefficients $u_j(z)$ consistently to any order.

The general expansion of a non-integrable equation will fail the Painlevé test at one of these two steps. Leading-order analysis for Equation (1) can be done by balancing the highest order derivative in t, namely $(D/2 - \mathrm{i}\beta)\ \psi_{tt}$ with the strongest nonlinearity, which is $(1-\mathrm{i}\varepsilon)\ \psi^2\bar{\psi}$. The integral term, clearly, does not contribute to this balance. This can be shown by comparing contributions from each term. Let ψ be a complex field and $\bar{\psi}$ be its complex conjugate. The above-mentioned balance can be written in the form

$$(\tfrac{1}{2}D - \mathrm{i}\beta)\ \psi_{tt} \sim -(1-\mathrm{i}\varepsilon)\psi^2\bar{\psi} \tag{2}$$

and

$$(\tfrac{1}{2}D + \mathrm{i}\beta)\bar{\psi}_{tt} \sim -(1+\mathrm{i}\varepsilon)\bar{\psi}^2\psi. \tag{3}$$

Substituting the leading-order *ansatz*

$$\psi \sim A_0\chi^p$$

and

$$\bar{\psi} \sim B_0\chi^{\bar{p}}$$

in the Equations (2, 3), where $\bar{p}$ is the conjugate of p and assuming $\chi_t \sim 1$, we obtain $p = -1 - \mathrm{i}d$. The constants d, A_0 and B_0 are the solutions of the following set of equations

$$D - \tfrac{1}{2}Dd^2 + 3\beta d = -A_0B_0, \tag{4}$$

$$\beta d^2 + \tfrac{3}{2}Dd - 2\beta = \varepsilon A_0B_0. \tag{5}$$

Hence,

$$d = \frac{3(D+2\varepsilon\beta) \pm \sqrt{9(D+2\varepsilon\beta)^2 + 8(\varepsilon D - 2\beta)^2}}{2(\varepsilon D - 2\beta)} \tag{6}$$

and the leading-order exponent $p = -1 - \mathrm{i}d$ is not an integer unless $d = 0$. This means that Equation (1) has already failed the Painlevé test at the first step.

The case $d = 0$ requires a special relation between the coefficients of the equation, *viz.* $\beta = \varepsilon D/2$, as can be seen from (6). However, in the latter case the recursion relations cannot be solved consistently. Let us show this. The integral term is very similar to that appearing in [5] and [9], so we can handle it by defining the real quantity

$$S = \varepsilon|\psi|^2 + \alpha \int_{-\infty}^{t} |\psi|^2 \,\mathrm{d}t'.$$

Then we get the following two equations which are equivalent to the original mCGLE

$$\begin{aligned} &\mathrm{i}\psi_z + \tfrac{1}{2}D(1-\mathrm{i}\varepsilon)\psi_{tt} + \psi[|\psi|^2 - \mathrm{i}(\delta + S)] = 0, \\ &S_t = \varepsilon(|\psi|^2)_t + \alpha|\psi|^2. \end{aligned} \tag{7}$$

We now set $\psi = a + \mathrm{i}b$, with a and b real. This leads to

$$\begin{aligned} &-b_z + \tfrac{1}{2}D(a_{tt} + \varepsilon b_{tt}) + a(a^2+b^2) + b(\delta + S) = 0, \\ &a_z + \tfrac{1}{2}D(b_{tt} - \varepsilon a_{tt}) + b(a^2+b^2) - a(\delta + S) = 0, \\ &S_t = 2\varepsilon(aa_t + bb_t) + \alpha(a^2+b^2). \end{aligned} \tag{8}$$

By setting $a = a_0\varphi^p$, $b = b_0\varphi^q$, $S = S_0\varphi^r$, we can equate leading coefficients, obtaining $p = q = -1$, $r = -2$. Hence, the leading terms are of order φ^{-3} and we can relate the coefficients.

The equation in S leads to

$$S_0 = \varepsilon(a_0^2 + b_0^2),$$

while the first two lead to

$$a_0^2 + b_0^2 = -D\varphi_t^2.$$

Thus, one of the functions a_0 and b_0 is arbitrary.

Now we can write a (3×3) matrix equation to find coefficients a_j, b_j and S_j. If we define $c = j(j-3)(a_0^2 + b_0^2)$ for convenience, then

$$n_{11} = 4a_0b_0 + c\varepsilon, n_{22} = 4a_0b_0 - c\varepsilon,$$

while

$$n_{12} = 4b_0^2 - c, \qquad n_{21} = 4a_0^2 - c.$$

The other elements are $n_{13} = -2a_0$, $n_{23} = 2b_0$, $n_{31} = 2\varepsilon a_0(2-j)$, $n_{32} = 2\varepsilon b_0(2-j)$, $n_{33} = j - 2$. The determinant of this matrix is

$$-(a_0^2 + b_0^2)^2(1+\varepsilon^2)(j+1)j(j-2)(j-3)(j-4),$$

so the resonances occur at $j = -1, 0, 2, 3, 4$.

Now we use the Laurent series

$$a = \frac{a_0}{\varphi} + a_1 + a_2\varphi + \cdots, \qquad b = \frac{b_0}{\varphi} + b_1 + b_2\varphi + \cdots,$$

and

$$S = \frac{S_0}{\varphi^2} + \frac{S_1}{\varphi} + S_2 + \cdots$$

Collecting terms of order $(\varphi^{-2}, \varphi^{-2}, \varphi^{-2})$ allows us to find S_1

$$S_1 = 2\varepsilon(a_0a_1 + b_0b_1) - \alpha D\varphi_t, \tag{9}$$

and also the equations relating a_1 and b_1 to a_0 and b_0. We find

$$a_1 = (k_1n_4 - k_2n_2)/P, \tag{10}$$

$$b_1 = (k_2n_1 - k_1n_3)/P, \tag{11}$$

where

$$n_1 = 2a_0b_0 - \varepsilon(3a_0^2 + b_0^2), \qquad n_2 = a_0^2 + 3b_0^2 - 2\varepsilon a_0b_0,$$

$$n_3 = 3a_0^2 + b_0^2 + 2\varepsilon a_0b_0, \qquad n_4 = 2a_0b_0 + \varepsilon(a_0^2 + 3b_0^2),$$

while

$$k_1 = a_0\varphi_z + D\varphi_t[b_{0t} + a_0\alpha - a_{0t}\varepsilon] + \tfrac{1}{2}D\varphi_{tt}(b_0 - \varepsilon a_0),$$

and

$$k_2 = -b_0\varphi_z + D\varphi_t[a_{0t} + b_0\alpha + b_{0t}\varepsilon] + \tfrac{1}{2}D\varphi_{tt}(a_0 + \varepsilon b_0).$$

Furthermore, P is the determinant

$$P = n_1n_4 - n_2n_3 = -3(a_0^2 + b_0^2)^2(1 + \varepsilon^2) \tag{12}$$

In the next level expansion ($j = 2$), the term S_2 does not appear in the third equation owing to the resonance at this value. This equation only involves S_{1t} and so we can simplify it using (9). This leads to

$$\alpha[2(a_0a_1 + b_0b_1) - D\varphi_{tt}] = 0.$$

Substituting in the known values of a_1 and b_1 from (10) and (11) shows that the part in brackets is not zero, so that we require $\alpha = 0$. Thus we are left with the CGLE, and Conte and Musette [6] have shown that it does not satisfy the Painlevé condition. Nevertheless, exact solutions still exist and can be found. We obtain one of them in the next section, using an original technique which we call 'segment energy and momentum balance'. We apply an *ansatz* similar to that in [7].

4. Balance equations in general form

Equation (1) does not have any conserved quantities. However, we can write balance equations related to the total energy $Q = \int_{-\infty}^{\infty} |\psi|^2 \, \mathrm{d}t$ of the solution

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = F[\psi], \tag{13}$$

where

$$F[\psi] = 2\int_{-\infty}^{\infty} \left[\delta|\psi|^2 + \varepsilon|\psi|^4 - \beta|\psi_t|^2 + \alpha|\psi|^2 \int_{-\infty}^{t} |\psi|^2 \, \mathrm{d}t' \right] \mathrm{d}t$$

and for its momentum $M = \Im \int_{-\infty}^{\infty} \psi\psi_t^* \, \mathrm{d}t$:

$$\frac{\mathrm{d}M}{\mathrm{d}z} = J[\psi], \tag{14}$$

where

$$J[\psi] = 2\Im \int_{-\infty}^{\infty} \left[\delta\psi + \varepsilon|\psi|^2\psi + \beta\psi_{tt} + \alpha\psi \int_{-\infty}^{t} |\psi|^2 \, \mathrm{d}t' \right] \psi_t^* \, \mathrm{d}t,$$

and the symbol $\Im$ indicates 'imaginary part of'. Clearly, these two equations can be used as replacements of the conservation laws in the case of Hamiltonian systems. For example, they are useful in finding stationary solutions because the left-hand sides of these equations are then zero and we have two functional equations:

$$F[\psi] = 0, \qquad J[\psi] = 0.$$

Generally speaking, the use of these allows us to find two parameters of the solution ψ. For example, if we are interested in bound states of two solitons, we can find the distance and the phase difference between the solitons using two balance equations [18]. Actually, a simple refinement of this technique allows us to find more parameters of the solution.

5. Segment energy and momentum balance

We suggest here a more general method which allows us to find, in certain cases, the solutions of (1). We are interested here in stationary solutions, and the pulse energy and momentum for them do not change with z. There must be an exact balance between the overall loss and gain from various sources for the whole pulse, but this must also apply within each segment of the pulse, since the soliton represents an equilibrium which is achieved because loss and gain effects counteract others. Similar considerations can be applied to the momentum. Equations similar to those in the previous section can be written for each segment in t. In what follows, we use them for analysing stationary solutions with nonzero velocity. Correspondingly, we derive these balance equations for stationary solutions rather than for general solutions.

First we convert to the moving group velocity frame by setting $\zeta = t - Vz$. Then, using the *ansatz*

$$\psi(\zeta, z) = f(\zeta)\exp[\mathrm{i}z(KV - \omega)],$$

we substitute it in (1). The resulting equation is

$$\tfrac{1}{2}Df'' - \mathrm{i}Vf' - (KV - \omega)f + |f|^2 f$$
$$= \mathrm{i}\delta f + \mathrm{i}\beta f'' + \mathrm{i}\varepsilon |f|^2 f + \mathrm{i}\alpha f \int_{-\infty}^{\zeta} |f|^2\, \mathrm{d}\zeta'. \tag{15}$$

Multiplying by f^*, taking the complex conjugate, subtracting the two expressions and integrating over ζ, we find that:

$$\tfrac{1}{2}DW(\zeta) - \tfrac{3}{2}\beta(|f|^2)' - \tfrac{1}{2}V|f|^2$$
$$= \delta \int |f|^2\, \mathrm{d}\zeta + \varepsilon \int |f|^4\, \mathrm{d}\zeta - \beta \int |f'(\zeta)|^2\, \mathrm{d}\zeta$$
$$+\alpha \int |f|^2 \left[\int_{-\infty}^{\zeta} |f(\zeta')|^2\, \mathrm{d}\zeta'\right] \mathrm{d}\zeta. \tag{16}$$

where $W = \Im(f' f^*)$. This equation is the consequence of the energy balance.

On the other hand, multiplying (15) by $f^{*\prime}$, taking the complex conjugate, and adding the two expressions, we find that

$$-(KV - \omega)(|f|^2)' + \tfrac{1}{2}D(|f'|^2)' + \tfrac{1}{2}(|f|^4)'$$
$$= 2\delta W + \mathrm{i}\beta(f'' f^{*\prime} - f^{*\prime\prime} f') + 2\varepsilon |f|^2 W + 2\alpha W \int_{-\infty}^{\zeta} |f(\zeta')|^2\, \mathrm{d}\zeta'. \tag{17}$$

Integrating with respect to ζ, we find

$$(\omega - KV)|f|^2 + \tfrac{1}{2}D|f'|^2 + \tfrac{1}{2}|f|^4 = 2\int W g(\zeta)\, \mathrm{d}\zeta - 2\beta\Im \int f'' f^{*\prime}\, \mathrm{d}\zeta \tag{18}$$

where

$$g(\zeta) = \delta + \varepsilon |f|^2 + \alpha \int_{-\infty}^{\zeta} |f(\zeta')|^2\, \mathrm{d}\zeta'.$$

This equation is the result of the balance of momentum.

We now represent the complex function f in terms of a single real function $A(\zeta)$. If we set $\chi = 1/A$ in the Painlevé analysis (Section 3), then we expect the leading order to be

$$f \sim A^{1+\mathrm{i}d} = A\ \exp(\mathrm{i}d\ \log\ A)$$

where A should involve the basic soliton sech function. Thus, using real parameters K and d, we have

$$f(\zeta) = A(\zeta)\ \exp[\mathrm{i}K\zeta + \mathrm{i}\, d\ \log(A(\zeta))].$$

This leads to $W = KA^2 + dAA'$ and two equations, one for energy

$$(\tfrac{1}{2}Dd - \beta)A(\zeta)A'(\zeta) + (\tfrac{1}{2}DK - \tfrac{1}{2}V + \mathrm{d}K\beta)A^2(\zeta)$$
$$= (\delta - \beta K^2)\int A^2\,\mathrm{d}\zeta + \varepsilon\int A^4\,\mathrm{d}\zeta - \beta(1+d^2)\int A'(\zeta)^2\,\mathrm{d}\zeta$$
$$+\alpha\int A^2\left[\int_{-\infty}^{\zeta} A^2(\zeta')\,\mathrm{d}\zeta'\right]\mathrm{d}\zeta, \tag{19}$$

and one for momentum

$$(\omega - KV)A^2(\zeta) + \tfrac{1}{2}D[(1+d^2)(A')^2 + K^2A^2 + 2\,\mathrm{d}KAA'] + \tfrac{1}{2}A^4$$

$$= 2\int A(\mathrm{d}A' + KA)\left(\delta + \varepsilon A^2 + \alpha\int_{-\infty}^{\zeta} A^2\,\mathrm{d}\zeta'\right)\mathrm{d}\zeta$$
$$-2\beta\int\left[(2+3d^2)KA'^2 + K^3A^2 + 3\,\mathrm{d}K^2AA' - KAA'' + d(1+d^2)\frac{A'^3}{A}\right]\mathrm{d}\zeta. \tag{20}$$

This is a set of integro-differential nonlinear equations involving the real function $A(\zeta)$. The general solution can be quite complicated, even for the class of the pulse-like solutions. We can try specific forms of the function A. Here we choose

$$A(\zeta) = \gamma C\,\mathrm{sech}(\gamma\zeta). \tag{21}$$

Then, substituting this in (19), we obtain an equation involving the functions $\tanh^n(\gamma\zeta), n = 1, 2, 3$. The whole expression must be identically zero, so we can equate the coefficients of each function to zero. For convenience, we can use 0 as a lower limit and allow an arbitrary upper limit for each term. This leads to 3 segment balance energy equations which constrain the solution parameters

$$(\beta - \tfrac{1}{2}Dd)\,\gamma^2 = \delta - \beta K^2 + (\alpha + \varepsilon\gamma)\gamma C^2, \tag{22}$$

$$V = \alpha C^2 + K(D + 2\beta d), \tag{23}$$

$$3(\beta - \tfrac{1}{2}Dd) = \varepsilon C^2 + \beta(1+d^2). \tag{24}$$

If we use integrals and evaluations over the range $(-\infty, \infty)$, then $W(\zeta = \pm\infty)$ is zero and we obtain only one equation for the parameters:

$$\beta(1+d^2)\gamma^2 = 3(\delta - \beta K^2) + (3\alpha + 2\varepsilon\gamma)\gamma C^2. \tag{25}$$

It is clear that this result can be derived from (22) and (24) above, so that some information is lost when the total-pulse-energy (*i.e.* we integrate from $\zeta = -\infty$ to ∞) equation is used. So these results constrain possible solutions and can be used to help solve the CGLE, as it does not have any of the conserved quantities which usually assist in solving such problems.

Using Equation (20) and the *ansatz* (21), we get further equations relating the solution parameters. This time we get five equations by separately equating coefficients of $\tanh^n(\gamma\zeta)$, $n = 1$ to 5. The $n = 5$ equation involves t_R only and thus indicates that this term must be zero for a solution of form (21). The other equations are

$$\delta + \alpha\gamma C^2 + \varepsilon\gamma^2 C^2 - \beta(K^2 + \gamma^2) + \tfrac{1}{2}D\,\mathrm{d}\gamma^2 = 0, \tag{26}$$

$$C^2[\alpha(K - \mathrm{d}\gamma) + \gamma^2(1 - \varepsilon\,\mathrm{d})] - \delta\,\mathrm{d} + K^2(3\beta d + \tfrac{1}{2}D)$$
$$+\omega - KV - \tfrac{1}{2}D(1 + d^2)\gamma^2 = 0, \tag{27}$$

$$3\,\mathrm{d}K(D + 2\beta\,\mathrm{d}) + 2C^2(K\varepsilon + \mathrm{d}\alpha) = 0, \tag{28}$$

$$C^2(\varepsilon\,\mathrm{d} - 1) + (1 + d^2)(D + \mathrm{d}\beta) = 0. \tag{29}$$

It turns out that Equation (26) is the same as Equation (22) above, but the other three are distinct. Thus, we have six independent equations which can be used to find the six parameters of the solution in terms of the parameters of the modified CGLE (1). To be specific, the unknown parameters of the solution are d, ω, K, γ, C and V.

6. Soliton parameters

We can now use the energy Equations (22)–(24) and the momentum Equations (26)–(29) to find the actual solution parameters. First, we combine (24) and (28) to obtain K as a function of C

$$K = -\frac{\alpha\,\mathrm{d}C^2}{2\beta(1 + d^2)}.$$

Next we use (23) to obtain V as a function of K

$$V = K\left(D - \frac{2\beta}{d}\right),$$

Continuing this process, we next obtain C in terms of the other parameters

$$C = \sqrt{\frac{3d(D^2 + 4\beta^2)}{2(2\beta - \varepsilon D)}}.$$

Then (22) gives γ directly:

$$\gamma = \gamma_\pm = \frac{\alpha C^2 \pm \sqrt{\alpha^2 C^4 - 2(2\beta - dD - 2C^2\varepsilon)(\beta K^2 - \delta)}}{2\beta - dD - 2C^2\varepsilon}, \tag{30}$$

Finally we use Equation (24) with C above to obtain d

$$d = \frac{3(D + 2\varepsilon\beta) - \sqrt{9(D + 2\varepsilon\beta)^2 + 8(\varepsilon D - 2\beta)^2}}{2(\varepsilon D - 2\beta)},$$

and then substitute (22) and (23) in (27) and simplify to get ω

$$\omega = \tfrac{1}{2}D(K^2 + \gamma^2) + \beta\, \mathrm{d}\gamma^2 - (\gamma C)^2.$$

This completes the process of finding the solution. It is clear that the form of d which we have found agrees with that found from the Painlevé analysis (*viz.* (6)). Hence the overall solution of Equation (1) has the form

$$\psi = A(\zeta)\, \mathrm{e}^{\mathrm{i}d\ \log[A(\zeta)]}\, \mathrm{e}^{\mathrm{i}Kt - \mathrm{i}\omega z}. \tag{31}$$

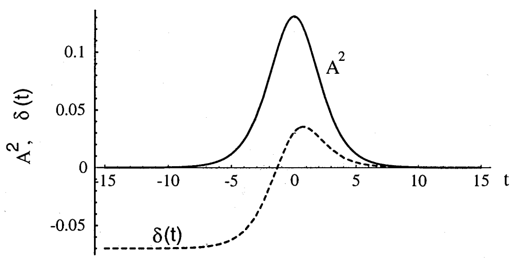

Figure 1. Soliton profile (solid line) and the loss curve $\delta(t) = \delta + \varepsilon|\psi|^2 + \alpha \int_{-\infty}^{t} |\psi|^2\, \mathrm{d}t$ (dotted line) defined by the exact solution (31) for $\varepsilon = 0{\cdot}1$, $\delta = -0{\cdot}05$, $\alpha = 0{\cdot}1$, $\beta = 0{\cdot}02$, and $D = +1$.

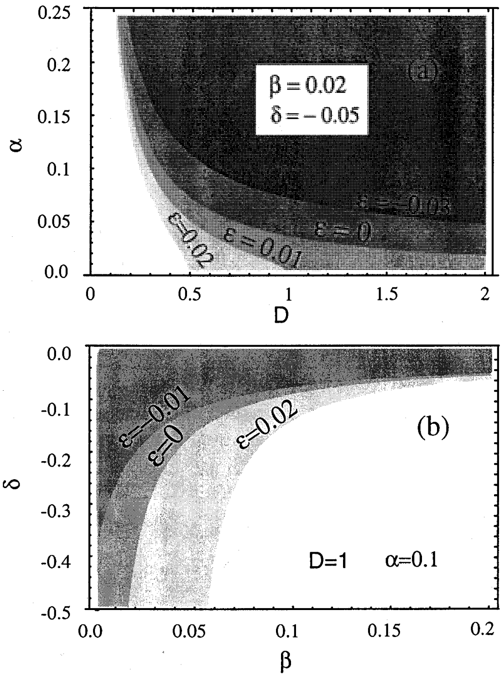

Figure 2. The space of parameters (a) α and D and (b) δ and β, where the soliton solution (31) exists. Shaded areas are defined by the inequality (32) for various values of ε and indicate where the solutions can exist. The parameters used in the calculation are shown in the plots.

We note that the form of d ensures that C is real and positive, because d and $(2\beta - \varepsilon D)$ have the same sign. The parameters $(\alpha, \beta, \delta, \varepsilon, D)$ must be chosen to ensure that the quantity under the square root sign in γ is positive. Substituting these solution parameters in (22)–(29) shows that the energy and momentum balance equations are indeed satisfied. Using the exact solution, we can easily calculate the total pulse energy

$$Q = 2|\gamma|C^2 = 3d|\gamma|\frac{D^2 + 4\beta^2}{2\beta - \varepsilon D}.$$

All the parameters of this solution, including the velocity V, are fixed and depend on the parameters of the equation. However, there are two branches of the solution, as specified by the two signs in (30).

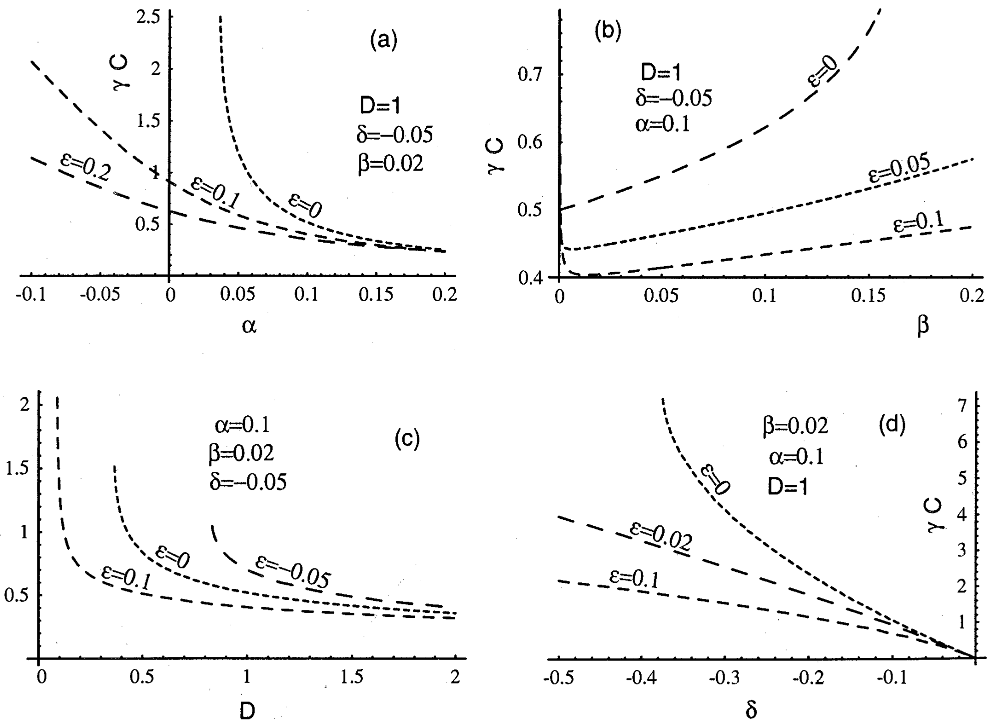

Figure 3. Dependence of the pulse amplitude, γC, on the parameters of the equation (a) α, (b) β, (c) D and (d) δ ($\gamma = \gamma_-$). The parameters used in the calculation are shown in the plots. Various curves correspond to various values of ε.

An example of the solution for certain values of parameters δ, β and α is shown in Figure 1. As can be seen from Figure 1, the soliton always clings to the gradient of the absorption curve $\delta(t)$. Different values of loss/gain on either side of the soliton can cause it to move relative to the reference frame. For $\gamma = \gamma_-$ the solution amplitude is close to zero when $\delta = 0$ (see Figure 3(b)). For $\gamma = \gamma_+$ it is not close to zero when $\delta = 0$. Note that δ is equal to the amount of loss (or gain) experienced by the left-hand side of the pulse.

The soliton exists for a certain range of parameters. The limits of existence are defined by the inequality

$$\alpha^2 C^4 - 2(2\beta - \mathrm{d}D - 2C^2\varepsilon)(\beta K^2 - \delta) > 0. \tag{32}$$

Figure 2 shows some regions where this inequality is valid, so that the solution (31) exists. First, consider the practical case where δ is negative. In this case, D can be either positive or negative. We now consider positive D and we choose $\gamma = \gamma_-$. The dependence of the soliton amplitude γC on α and δ is given in Figure 3. An important parameter for chirped pulses is the amplitude-width product C. It does not depend on α or δ, but depends weakly on β and D as shown in Figure 4. The velocity V of the soliton does not depend directly on δ, but depends linearly on α because K depends linearly on α. The velocity varies with β and D as shown in Figure 5 and can be positive, negative or zero. In the case of non-negative ε the velocity is always positive.

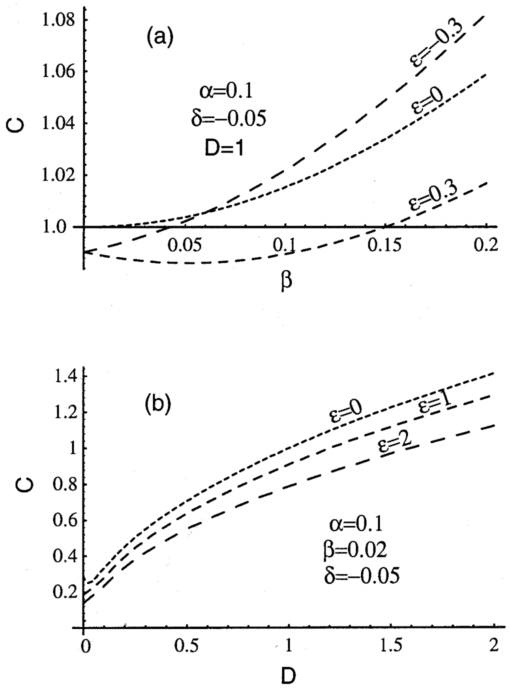

Figure 4. Amplitude-width product C of the soliton versus (a) β and (b) D. The parameters used in the calculation are shown in the plots. Various curves correspond to various values of ε.

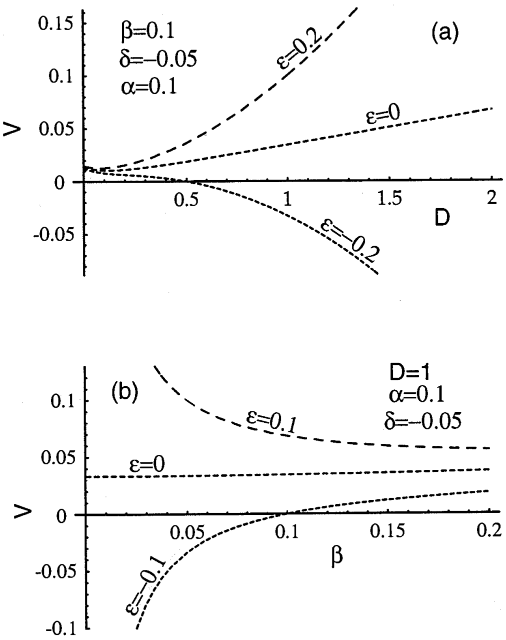

Figure 5. Soliton velocity V versus (a) D and (b) β. The parameters used in the calculation are shown in the plots. Various curves correspond to various values of ε.

In the case of δ positive, the solution exists for each sign in the expression for γ. Hence, we have simultaneously two solutions for the same set of parameters. The dependence of the soliton amplitude γC on α and δ for one of the branches is shown in Figure 6. Moreover, the solution exists for both normal and anomalous dispersion (negative and positive D). This is not surprising [18], because in systems with gain and loss, the pulse is the result of a balance not only of the dispersion and nonlinearity (which is impossible at negative D), but also of gain and loss. The latter balance can be much stronger than the former one and can have a decisive influence on the soliton formation.

If we take $D > 0$, then the limit $\beta \to \frac{1}{2}\varepsilon D$ provides the chirp-free ($d = 0$) case:

$$K = 0, \qquad V = D\alpha, \qquad C = \sqrt{D}, \qquad \omega = -\tfrac{1}{2}D\gamma^2,$$

$$\gamma = \gamma_\pm = \frac{-\alpha \mp \sqrt{\alpha^2 - 2\varepsilon\,\frac{\delta}{D}}}{\varepsilon}. \tag{33}$$

This special case of the general solution can be useful when analysing generation of chirp-less pulses by passively mode-locked lasers.

One of the most important properties of the pulses is their stability. In our case, the background is unstable if there is positive gain on one or both sides of the pulse. Therefore δ must be negative, as this causes the total gain to be negative on the l.h.s. of the pulse. Total gain can be positive, negative or zero on the right-hand side of the pulse. In the negative case only, this region may be stable with respect to both generation of continuum and new pulses. Then

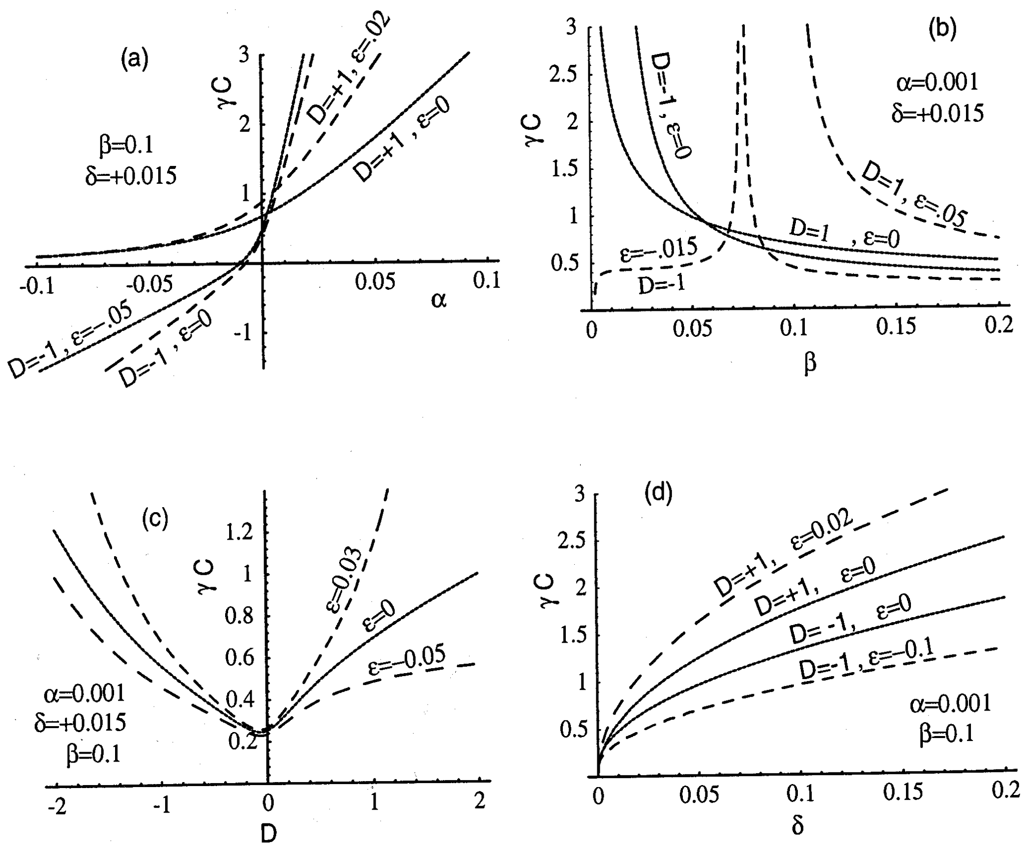

Figure 6. Dependence of the pulse amplitude, γC, on the parameters of the equation (a) α, (b) β, (c) D and (d) δ ($\gamma = \gamma_+$). The parameters used in the calculation are shown in the plots. Various curves correspond to various values of ε and D.

the stability of the solution is defined completely by the stability of the pulse itself. Even in this case the pulse is unstable when the parameters in Equation (1) are fixed. Any increase of the amplitude of the pulse relative to the exact solution increases the total gain across the pulse and the amplitude increases exponentially. Any decrease of the pulse amplitude works in the opposite direction and the pulse decays. However, for a proper choice of parameters, the pulse may become stable if δ depends on the total energy of the pulse Q. The $\delta(Q)$-dependence exists in passively mode-locked lasers and it may serve as a feedback mechanism which stabilizes the pulse for a certain range of parameters in the equation.

Conclusions

In conclusion, we have analysed the modified complex Ginzburg-Landau equation. It includes a delayed response term in the integral form which describes, for example, passively mode-locked laser systems with slow saturable absorbers. We presented a singularity analysis of the mCGLE in order to identify the integrable models. It turned out that this equation with nonzero nonconservative terms fails to pass the Painlevé test. Nevertheless, exact solutions to this equation do exist. We have shown that its stationary solutions can be found using balance equations. These are extensions of conservation laws to nonconservative systems. As a particular example, using them, it is possible for us to derive the soliton solution of the modified CGLE. The same technique can be used to obtain the solutions of the cubic-quintic CGLE, and the CGLE with Raman [20] and self-steepening terms in it. A number of other cases can also be analysed. An advantage of using the method of balance equations is that not only one-soliton solutions can be analysed in this way, but also multi-soliton solutions [18] which usually cannot be treated as easily with other methods.

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